STEM
RESOURCES

MATHEMATICS

> Calculus
> Algebra

ferrovial

INDEX

Calculus

Algebra

CALCULUS

What is calculus?

In the most general sense, calculus is the act of calculating - that is, **following a procedure to obtain the result of an operation.** From a more mathematical approach, it explains the algorithmic process to get the result of symbolized variables that are known in advance.

This discipline follows **a structure of steps** based on knowing certain data whose numerical value may or may not be known. In addition, it offers a response to the **arithmetic operations of addition, subtraction, multiplication, and division**, though unlike algebra: it draws on the same procedures but does so in a more abstract way, substituting numbers for letters.

Calculus is also useful for **determining areas, volumes, curves, slopes, and the maximum and minimum values of a function,** which is why it is closely linked to other scientific disciplines such as mathematics, physics, and engineering.

What terms are related to mathematical calculation?

- **Function:** the relationship between two sets, where the first set is given a single element of the second, meaning its value depends on that given.
- **Variables:** amounts that can have an indefinite number of changes generated by different conditions.
- **Constants:** Unlike variables, a constant has a fixed value that does not change.
- **Derivatives:** the term used to calculate the answer to a function by altering its initial value.
- **Increment:** the difference between two values of a variable.
- **Limit:** this term is attributed to a variable when it approaches a constant value without becoming equal to it, with the value changing according to a set of factors.

What are the branches of calculus?

There are different forms of calculus, each with its own characteristics, structure, and theorems. The main branches include:

- **Infinitesimal calculus:** the study of limits, derivatives, integrals, and infinite series.
- **Differential calculus:** establishes the change of an object according to its variables and focuses mainly on the study of movement, speed, and acceleration.
- **Integral calculus:** calculates a value when the acceleration in the area where it is located is known.

How is calculus applied in other fields or disciplines?

Mathematical calculus explains many phenomena and processes that surround humankind. **Modern technology would not be possible without this discipline**, which can be applied to nature, as well as to economics, administration, and physics. Basically, **every scientific approach draws on equations that, in turn, involve functions and derivatives, and their analysis is done by using calculus tools.**

For example, in **engineering**, calculus makes it possible to measure structures and areas; in **environmental** fields, calculus can be used to count and study the growth of various organisms; in **electronics**, it is used to calculate current loads or discharges, as well as their times; in **accounting**, it makes it possible to determine the profits or losses that may be generated by different projects.

CALCULUS

>COMPLEX NUMBERS

What are complex numbers?

Complex numbers are the **combination of real and imaginary numbers.** The real part can be expressed by an integer or decimal, while the imaginary part has a square that is negative. Complex numbers arise from the need to **express negative numbers' roots,** which real numbers can't do. This is why **they reflect all the roots of polynomials.**

Their use extends to different scientific branches, ranging from **mathematics to engineering.** Complex numbers can also **represent electromagnetic waves and electric currents,** so they are essential in the field of electronics and telecommunications.

Its mathematical formula is **a + b i**, where **a and b are real numbers, and i is the imaginary number.** This expression is known as **binomial form** because of the two parts that make it up.

What is the origin of complex numbers?

French mathematician **René Descartes** was the first to emphasize the imaginary nature of numbers, positing that "one can imagine as many (numbers) as already mentioned in each equation, but sometimes, there is no quantity that matches what we imagine."

However, **the conceptualization of complex numbers dates back to the 16th century** with the contribution of Italian mathematician Gerolamo Cardano, who **proved that having a negative term inside a square root can lead to the solution of an equation.** Up until then, it was thought to be impossible to find the square root of a negative number.

Later, in the 18th century, mathematician **Carl Friedrich Gauss** consolidated Cardano's premises, in addition to **developing a treatise on complex numbers in a plane and** thereby established the modern bases of the term.

What are the main characteristics of complex numbers?

- The real numbers involved in a complex number formula can be expressed in the form of an **ordered pair, a binomial, and a vector.**
- The whole set of imaginary numbers **is called i and is the equivalent of 1** in the real numbers. Similarly, the square root of **i** is -1.
- Two complex numbers are considered equal when they have the same real and imaginary components.
- The letter **C represents the set of all complex numbers. C also forms a two-dimensional vector space.**
- Unlike real numbers, **complex numbers have no natural order.**
- There are pure imaginary numbers, the real part of which is 0; their formula is as follows: 0 + bi = bi.

What is the importance of complex numbers?

While their day-to-day application is not as direct as that of real numbers, their imaginary component makes complex numbers **important as they make it possible to work very precisely in specific areas of science and physics**. This is the case with measuring electromagnetic fields, which consist of electrical and magnetic components and require pairs of real numbers to describe them. These pairs can be seen as a complex number, hence their importance.

How are complex numbers represented graphically?

Any numerical category (whether natural, integer, or rational) can be represented graphically on a line. In the case of real numbers, they cover the line completely, and every number corresponds to a place on the line (also called the real line).

Complex numbers leave the line to fill a plane called the complex plane. In this case, **complex numbers are represented on Cartesian axes,** where the **X** axis is called the **real axis** and **Y** the **imaginary axis**. The formula for complex numbers, a + bi, is represented by the point or end (a,b), called the affix, or by a vector with the origin (0,0).

CALCULUS

>TRIGONOMETRY

What is trigonometry?

Trigonometry is the **area of mathematics** dealing with the **study and measure of triangles, the relationships between their angles and sides, and the trigonometric functions** of sine, cosine, tangent, cotangent, secant, and cosecant.

This mathematical branch is related to other scientific areas directly and indirectly, and **it applies to everything that requires precision measurements**, such as **spatial geometry and astronomy,** for measuring the distances between stars with respect to other geographical points.

The study of trigonometry has existed for more than three thousand years. In Babylon and Egypt, the measurements for triangles' angles were used to build pyramids.

What is the importance of trigonometry?

The application of trigonometric functions in physics, astronomy, telecommunications, nautical fields, engineering, cartography, and others is what makes them significant: **they allow you to calculate distances accurately without necessarily having to travel them.**

Knowing this, the importance of trigonometry lies in the **various applications** it has in the following examples:

- Calculating the distance between two points where one, or even both, are inaccessible.
- Accurately calculating distances and angles of inclination, which is very useful for civil engineering.
- Calculating the height of a point that may also be inaccessible.

What are the units of measurement in trigonometry?

When studying the measurement of angles, as well as their calculation, trigonometry uses the following units:

1. **Radian:** basic angular unit in trigonometry that indicates the relationship that exists between the angle formed based on the radius of a circumference and an arc of the same length. A full circle is made up of two radians.

2. **Sexagesimal degree:** angular unit that divides a circumference into 360 degrees, considering that each right angle has 90 degrees, and if the circumference is divided into four equal parts, the sum of each angle will give a total of 360. It is usually used in the practical field of branches such as engineering, architecture, or physics.

3. **Centesimal degree:** an angular unit that divides a circumference into 400 centesimal degrees.

4. **Milliradian:** a unit that divides the circumference into 6,400 units.

What are the trigonometric functions?

Trigonometric functions are the **metric ratios between the sides of a right triangle.** For a triangle that has a right angle of 90 degrees, three fundamental elements can be determined:

1. **Angles:** the area of the plane that lies between two semi-straight lines with a common origin. This is the amplitude of the arc of a circle, with its center at the vertex and bounded by its sides.

2. **Legs:** the other sides that make up a triangle. They can be classified into the opposite leg (which is located on the opposite side or in front of the angle in question) and adjacent (which is located next to the angle in question).

3. **Hypotenuse:** the longest side of a triangle that is opposite the right angle.

Understanding these three fundamental concepts, **the trigonometric functions are:**

- **Sine:** the ratio between the opposite leg of the angle in question and the hypotenuse.

- **Cosine:** division of the adjacent leg of the angle in question by the hypotenuse of the triangle.

- **Tangent:** ration between the opposite side and the adjacent leg of the triangle. It is expressed as sine over cosine.

Every trigonometric function has its reciprocal ratio, i.e.:

- **Secant:** reciprocal ratio of the cosine consisting of the ratio between the length of the hypotenuse and the length of the adjacent leg.
- **Cosecant:** reciprocal ratio of the sine consisting of the ratio between the length of the hypotenuse and the length of the opposite leg.
- **Cotangent:** reciprocal ratio of the tangent consisting of the ratio between the length of the adjacent leg and that of the opposite one.

CALCULUS

>MATRICES

What are matrices?

The matrices are a **two-dimensional set of numbers or symbols** distributed in a rectangular shape in vertical and horizontal lines so that **their elements are arranged in rows and columns.** They are useful for **describing systems of linear or differential equations,** as well as representing a linear application.

Every matrix is represented by a **capital letter,** and its elements are given in lower-case letters in a list enclosed by parentheses or square brackets. Each, in turn, has a double superscript: the first refers to the row and the second to the column to which it belongs.

This mathematical expression can be added, multiplied, and decomposed, so **it is commonly used in** linear algebra**.**

What concepts are associated with matrices?

Some of the concepts needed to complete the definition and analysis of matrices are:

- **Elements:** the numbers that make up the matrix.
- **Dimension:** the result of the number of rows times the number of columns. The letter **m** is used to designate the number of rows and **n** for the number of columns.
- **Rings**: this algebra term refers to the system formed by a set of internal operations that respond to a set of properties. Matrices are understood as elements of a ring.
- **Function:** a correspondence rule between two sets in which an element of the first set corresponds exclusively to a single element of the second set.

What types of matrices are there?

A matrix can be:

1. **Rectangular:** it has different numbers of rows and columns.
2. **Row:** a rectangular array with a single row.
3. **Column:** a rectangular matrix with a single column.
4. **Null:** an array that has zero elements.
5. **Square of order n:** a matrix that has the same number of rows as columns. In this type of matrix, the dimension is called the **order**, and its value coincides with the number of rows and columns.
6. **Diagonal:** a kind of square matrix where the elements not located on the main diagonal are equal to zero.
7. **Scalar:** a diagonal matrix where all the elements on the main diagonal are equal.
8. **Identity:** this is a scalar matrix where the elements of the main diagonal are equal to one, while all other elements are equal to zero.
9. **Inverse:** the opposite of another matrix whose elements have signs opposite to the main matrix. That is, the inverse matrix of A is called -A, and all the elements of the set are the opposites of the elements of matrix A.
10. **Transpose:** the matrix obtained when converting rows to columns. The superscript **t** is used to represent it, and its dimension is **n** x **m**.
11. **Upper triangular:** this is a square matrix where at least one of the terms above the main diagonal is non-zero, and all those below the main diagonal are equal to zero.
12. **Lower triangular:** unlike the previous type, this type of matrix has at least one element below the main diagonal that is non-zero, and all those above the main diagonal are equal to zero.

How can matrices be used?

Matrices have multiple applications, especially for **representing coefficients in systems of equations or linear applications**; **a matrix can perform the same function as vector data** in a linear system of application. Depending on this, some applications include:

1. **In computer science:** one of the fields where matrices are most used, given their effectiveness in working with information. Matrices are **ideal for graphic representations** and **animating** shapes.

2. **In robotics:** matrices are used for **programming robots** that can execute different tasks. One example of this is a **bionic arm** that can use programmable mechanical processes to fulfill functions similar to those of a human arm. All of this programming is the result of calculations using matrices.

CALCULUS

>PROBABILITY

What is probability?

The term probability is used to define the **mathematical calculation that establishes all the possibilities that exist for a phenomenon to occur in certain random circumstances.** Probability is calculated based on **a value between 0 and 1**, and the level of certainty is determined by the closeness to the unit value; on the other hand, if it is closer to zero, there is less certainty in the final result.

What is the formula for calculating probability?

To calculate probability, you must **divide the number of favorable events by the total number of possible events.** This generates a sample, and the calculation can be performed from the data obtained.

Calculating probabilities **is expressed as a percent** and follows the formula:
Probability = Favorable cases / possible cases x 100.

What types of probability are there?

- **Mathematical:** this follows the principles of formal, non-experimental logic, calculating random events that may occur within a certain field in figures.
- **Frequency:** based on experimentation and determines the number of times an event may occur by considering a specific number of opportunities.
- **Objective:** considers the frequency of the event in advance and only sheds light on the probable cases when that event may occur.
- **Subjective:** this concept is the opposite of mathematical probability, as it takes certain eventualities into account that allow inferring the probability of a certain event, even without having certainty at the arithmetic level.
- **Binomial:** determines the success or failure of an event with only two possible outcomes.
- **Logical:** raises the possibility of an event occurring based on inductive laws.
- **Conditional:** explains the probability of one event happening based on the prior occurrence of another, so one is dependent on the other.

- **Hypergeometric:** probability obtained from sampling techniques - that is, events are classified according to the frequency of their occurrence. This way, a set of groups of events are created that are determined according to their occurrence.

What theories explain probability?

There are three methods for determining the probability of any event, and they are based on the rules of:

1. **Addition:** states that the probability of a particular event occurring is equal to the sum of the individual probabilities, as long as the events do not occur at the same time.

2. **Multiplication:** posits that the probability of two or more independent events occurring is equal to the product of their individual probabilities.

3. **Binomial distribution:** posits that the probability of a given combination of events occurring independently of each other admits only two possible mutually exclusive outcomes: success or failure.

There is also **Laplace's rule**, which states that, in a random sample composed of results that are equally probable, the probability of an event is the result of the number of possible cases divided by the number of probable cases.

In what situations can probability be used?

Some examples where probability is applied are:

1. **Statistical analysis of business risk:** drops in stock prices, investment statements, etc. can be estimated through probabilistic formulas.
2. **Insurance calculation:** the processes used to study the reliability of an insured party, making it possible to know whether it is profitable to insure them and at what price and time span this should be done, arise from probability calculations and strategies.
3. **Behavioral analysis:** in this type of application, probability is used to evaluate certain behaviors of a population sample so that certain patterns of opinions, behaviors, or thoughts can be predicted.
4. **Medical research:** the success of vaccines, as well as their side effects in a population, is an example that's determined by probabilistic calculations.

ALGEBRA

What is algebra?

Algebra is the **branch of mathematics** that studies the combination of elements such as **numbers, letters, and signs** to create different **elementary arithmetic operations**. Algebra differs from arithmetic because it uses letters as abstractions to **represent unknown variables or quantities.** It also lets you write formulas **(algebraic formulas)** that **express a rule or principle and which make it easier to solve equations.**

The term comes from the treatise "The Compendious Book on Calculation by Completion and Balancing" by Muhammad ibn Musa al-Jwarizmi**,** a Persian mathematician who developed symbolic operations to solve equations systematically.

What is algebraic language?

In the study of algebra, there are **different terms for expressing the language of operations.** Among them are:

1. **Algebraic term:** this is the simple expression of a combination of letters and numbers, with no addition or subtraction. It is composed of: a sign, which can be positive or negative; a coefficient, which is the number that accompanies the variable; the variable, which is the unknown; and the exponent, which represents the power to which the variable is raised.

2. **Algebraic expression:** this consists of a set of numbers and variables that can be combined with different arithmetic operations. It can be composed of a single algebraic term (monomial) or up to more than three terms (polynomial).

3. **Algebraic equations:** this is the association of two algebraic expressions by an equal sign. They can be first-degree, when the variable is raised to the power of 1, or second-degree, when the variable is raised to power 2, also called a quadratic equation.

What are the origins of algebra?

The first indications of algebraic operations are found in Babylonian mathematics, which used pre-calculated tables to formulate and solve equations. These models were always positive, as they solved only real problems.

Greek mathematicians, on the other hand, developed geometric algebra. Diophantus of Alexandria is considered **the father of algebra**. His book, **Arithmetica**, is among the highest-level ancient arithmetic books, though only the first six of thirteen books have survived to today.

The first time zero and the possibility of negative numbers were considered was in the book **Brahmasphutasiddhanta**, by the Indian mathematician and astronomer Brahmagupta. Later, algebra was developed to more complex levels by the Arabs. The Persians are particularly noteworthy: Al-Juarismi, whose transliterated name gave birth to the word **algorithm,** and Omar Jayam, the creator of the concept of a **function**.

What is linear algebra, and how is it used?

Linear algebra is the branch of mathematics that **focuses on vectors, matrices, systems of linear equations, and dual space.** It is used in most sciences: from civil engineering, enabling modeling and computing all sorts of structures **to determine materials, shapes, and strengths of the constructions**; to **computing**, by making image processing, web searches, and image processing for video games and films possible; and even code manipulation to **optimize Machine Learning algorithms.**

ALGEBRA

>LINEAR ALGEBRA

What is linear algebra?

Linear algebra is the branch of algebra dealing with the **study of matrices, vectors, vector spaces, and linear equations.** These are mathematical functions that occur between vectors within linearity conditions or the set of successions that are a proportional outcome of a cause.

This type of algebra is a fundamental area within mathematics, especially in the field of geometry. **It lets you define objects such as lines, planes, and rotations.** It is also indispensable in the field of engineering because **it makes it possible to calculate, model, and compute natural phenomena.**

What are the elements of linear algebra?

In a linear algebra equation, whose **graphical representation is a straight line,** there are a number of elements that must be taken into account to solve it:

- **Vectors:** straight lines that indicate a defined direction and that are projected in a certain space. They are lines with points of origin, magnitudes, directions, coordinates, and lengths. They have a rectilinear graphic representation.
- **Graders:** elements used to describe a phenomenon with a magnitude but no direction. They can be a real, complex, or constant numbers.
- **Root:** the amount multiplied by itself as many times as necessary to get another amount as a result. The goal of the root is to get the base of the power by knowing the exponent and the subradical quantity.
- **Matrix:** a two-dimensional set of numbers arranged in rows and columns that allow for representing coefficients present in systems of linear equations.
- **Determinant:** the mathematical expression that results from the application of the elements in a square matrix.

How has algebra evolved throughout history?

The introduction of linear algebra in the West dates back to the year 1637, when René Descartes **develop the concept of coordinates** under a geometric approach, known today as Cartesian geometry. This concept proposes **representing the**

lines and planes through linear equations, making it possible to calculate their intersections through systems of linear equations.

German mathematician Gottfried Leibniz established **the use of determinants to solve linear systems** in 1693. In 1750, Swiss mathematician Gabriel Cramer used this concept **to solve linear systems and develop what is now known as Cramer's rule.**

Linear algebra, as it is known today, has been developed as a succession of contributions by scientists who continue to add terms. The contributions began in the year 1843 when Irish scientist William Rowan Hamilton developed the term **vector** and created **quaternions**. Quaternions are an extension of the real numbers to which imaginary units **i, j,** and **k** are added. They are based on complex numbers, which add the imaginary unit **i** to the set of real numbers.

A year later, German physicist Hermann Grassmann published the book **The Extension Theory,** developing topics and fundamental elements of this branch of algebra. Finally, in 1848, English mathematician James Joseph Sylvester added the term **matrix**.

What are the main characteristics of linear algebra?

- It looks at **vector spaces,** that is, mathematical structures where you can add between different elements (called vectors) of a set and multiply them by real or complex numbers.
- It is based on **systems of linear equations** with constants (numbers) and unknown information that is represented without exponents.
- It uses **letters and symbols to replace numbers** in arithmetic operations; these are known as **variables.**
- It is called **linear** because **the equation represents a straight line in the Cartesian plane.**
- It allows us to solve problems through **logical and mathematical tools** that can be applied to different sciences and branches of studies, but also for day-to-day situations.

ALGEBRA

>ABSTRACT ALGEBRA

What is abstract algebra?

Abstract algebra is the branch of algebra dealing with the **study of algebraic systems or structures with one or more mathematical operations** associated with elements with an identifiable pattern, differing from the usual number systems. In abstract algebra, the elements combined to perform mathematical operations **are not interpretable as numbers,** hence its abstract nature.

The elements of abstract algebra operate as an abstraction of the algebraic properties common to different number systems and other objects of mathematical study. Therefore, its objective is to **learn about the properties of operations**, regardless of the operands' characteristics. Most of this branch was created in the 19th century to respond to the need for greater accuracy in mathematical definitions.

What can abstract algebra be used for?

The main purpose of abstract algebra is **analyzing a set endowed with one or more operations with special characteristics or properties** to learn about the relationships between those properties of the operations in a precise way, as well as the consequences and possible results of their associations.

What are algebraic structures?

In abstract algebra, "structure" means a **set or group with one or more algebraic operations.** These structures are classified according to the number of operations that may exist in them, as well as by their characteristics, the number of elements or subsets, and the relationship that exists among the elements of the main set, **regardless of their nature.**

What is the Law of Composition?

This is a term from abstract algebra used to name a type of **binary operation** where two elements of given sets are assigned to another element, giving rise to distinct **algebraic structures**. The law of composition can be **internal or external**, depending on whether the elements in the application are part of the same set or of different sets, respectively.

The internal laws of composition are represented by the following symbols: ⊙, ◎, ⊕, ⊖, ⊛, ⊗, and ⊘. The **external laws of composition** are represented by the following symbols: ·, ∘, +, -, ⋇.x, and ⁄. The **sets** are represented by capital letters (A, B, C...) and their **elements** with lowercase letters (a, b, c...).

What types of algebraic structures are there?

The most common algebraic structures from the abstract branch can be classified as:

1. **A single law of composition or binary operation:**
   - **Magmas:** algebraic structures of the form (A,◎) where A is a set with a single internal binary operation.
   - **Semigroups:** structures of the form (A,◎) where A is a non-empty set and ◎ is an internal operation defined in A.
   - **Groups:** algebraic structures formed by a non-empty set with an internal operation that combines a pair of elements to compose a third element within a set.
   - **Quasigroups:** algebraic structures with a linear term that are configured like a magma with a single law of internal composition whose elements are divisible. Their main difference from groups is that they are not necessarily associative.
   - **Monoids:** algebraic structures with associative operations and a neutral element; the latter is what differentiates them from semigroups.

2. **Has two or more laws of composition or binary operations:**
   - **Rings:** algebraic systems composed of a set and two internal binary operations that are expressed as (R,+,•).
   - **Bodies** (also called **fields**)**:** they are commutative rings of division.
   - **Modules:** algebraic structures that act in group representation theory, where a group entails concrete transformations of a mathematical object.
   - **Vector spaces** (also called **linear spaces**)**:** algebraic structures composed of a non-empty set, an internal operation (called a sum), and an external operation (called a scalar product). The elements of a vector space are called vectors.
   - **Associative algebras:** they are modules that also allow the multiplication of vectors in a distributive and associative way.
   - **Lie algebras:** algebraic structures defined on a vector space and normally associated with the **Lie groups.** This mathematical object was previously called an **infinitesimal group.**

- **Lattices:** they are algebraic structures used in algebra and order theory; their name comes from the shape of **Hasse diagrams**.
- **Boolean algebras:** algebraic structures that outline logical operations, used both in mathematics and in digital electronics and computer science.

What is the difference between abstract algebra and elementary algebra?

While elementary and abstract algebra both respond to the same general approaches to algebra, there are differences between them; for example, while elementary algebra is based on solving simple algebraic equations, abstract algebra **looks at algebraic systems and structures** or groups with different operations. Elementary algebra studies real numbers and complex numbers, while abstract algebra expresses mathematical structures that cannot necessarily be expressed with numerical values.